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Absence of Second-Order Phase Transitions in the Dobrushin Uniqueness Region

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The Dobrushin uniqueness theorem assures that in a very large class of high-temperature classical statistical mechanical lattice models with short or long range, many-body interactions, and arbitrary compact spin space there are no first-order phase transitions. It will be shown that for the same class of interactions there are also no second-order phase transitions.

KEY WORDS: Classical lattice gases; second-order phase transitions; Dobrushin uniqueness theorem.

1. INTRODUCTION

Dobrushin's uniqueness theorem[1] asserts the uniqueness of Gibbs states for a very general class of long-range, many-body potentials in classical lattice models with arbitrary spin at high temperature. In a translation invariant theory the uniqueness theorem therefore assures the nonexistence of first-order phase transitions for these potentials and temperatures. In this paper we shall show that for the same potentials and temperatures there are, in fact, no second-order phase transitions either.

More explicitly, we shall show that the pressure is twice continuously differentiable in the entire Dobrushin uniqueness region, which may be regarded for these purposes as a subset of a naturally associated Banach space of potentials. The techniques are adapted from Ref. 6.

Gallavotti and Miracle-Sole have shown[5] that if the single-site spin space consists of just two points then qualitatively better results can be gotten by other techniques. They show, in particular, that the pressure is

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analytic at high temperature for a somewhat broader class of potentials than that to which the Dobrushin uniqueness theorem applies. This should be contrasted with results of the present paper which only show twice differentiability. Their techniques, however, seem to be limited to spin spaces consisting of a finite number of points. Thus for a system with a continuous spin space, such as a crystal with a diatomic molecule at each site, the only high-temperature technique that applies with great generality is Dobrushin's. It is not known whether analyticity of the pressure can actually fail at high temperature in the very large space of interactions to which Dobrushin's uniqueness theorem applies. For results on high-temperature analyticity with continuous spin space and a smaller, but quite general, space of many-body long-range interactions we refer the reader to Ref. 9. For general results on analyticity via high-temperature expansions we refer the reader to Refs. 3, 4, 7, and 8.

Because the proof has many technical details I have included a section (Section 3) which sketches the proof in an informal way. It includes a series expansion for the second derivative which is different from the usual series of covariances and which may conceivably be more useful for computations. In fact, although the usual series of covariances was treated in Ref. 6 by related techniques, I was not able to show that it converges in the entire Dobrushin uniqueness domain nor that it actually represents the second derivative of the pressure.

2. NOTATION AND STATEMENT OF RESULTS

Consider a classical statistical mechanical system on an m-dimensional cubic lattice Z^m with single spin space X. We take X to be a compact metric space. A many-body interaction is given by a real-valued function, φ, on the union of all products X^A wherein A runs over all nonempty finite subsets of Z^m. We assume $\varphi | X^A$ is continuous for each A and that φ is translation invariant. The energy of that portion of the system which lies in a finite subset Λ of Z^m is given by

$$U_\Lambda(s) = \sum_{A \subset \Lambda} \varphi(s | A)$$

when the configuration in Λ is s. Here s is in X^Λ and the sum runs over all nonempty subsets of Λ.

Denote by ν a finite measure on X, the *a priori* single-spin measure. Writing $|A|$ = number of points in A we put

$$\|\varphi\| = \sum_{A \ni 0} |A| \sup\{ |\varphi(s)| : s \in X^A \} \tag{2.1}$$

and denote by $\mathcal{P}$ the Banach space of translation invariant potentials for which $\|\varphi\| < \infty$.

The pressure, $P(\varphi)$, is defined, as usual, by

$$P(\varphi) = \lim_{\Lambda \to Z^m} |\Lambda|^{-1} \log \int_{X^\Lambda} e^{-U_\Lambda(s)} \nu^\Lambda(ds)$$

where $\nu^\Lambda = \prod_{a \in \Lambda} \nu$. As is well known (Ref. 13, Chap. 2), the limit exists for all φ in $\mathcal{P}$ when $\lim_{\Lambda \to Z^m}$ is suitably interpreted.

Dobrushin's uniqueness theorem asserts that φ has a unique Gibbs state, i.e., a unique solution to the DLR equations, for φ in a neighborhood, $\mathcal{D}$, of the origin in $\mathcal{P}$. $\mathcal{D}$ may be described as follows.

Let $\Omega = X^{Z^m}$. For each point a in Z^m put

$$W_a(s) = \sum_{A \ni a} \varphi(s \mid A), \qquad s \in \Omega \tag{2.2}$$

Since the series converges uniformly, W_a is continuous. Let

$$Z_a(t) = \int_X e^{-W_a(x \vee t)} \nu(dx), \qquad t \in X^{Z^m - \{a\}} \tag{2.3}$$

and

$$\mu_a(dx \mid s) = Z_a(t)^{-1} e^{-W_a(x \vee t)} \nu(dx) \tag{2.4}$$

where $t = s \mid Z^m - \{a\}$. For $b \neq 0$ in Z^m put

$$\rho_b = (1/2) \sup\{ \| \mu_0(\cdot \mid s) - \mu_0(\cdot \mid s') \|_{\mathrm{Var}} : s = s' \text{ off } \{b\} \} \tag{2.5}$$

and put

$$\alpha(\varphi) = \sum_{a \neq 0} \rho_a \tag{2.6}$$

The Dobrushin uniqueness region is

$$\mathcal{D} = \{ \varphi \in \mathcal{P} : \alpha(\varphi) < 1 \}$$

We shall show in Proposition 2 in Section 4 that $\alpha(\cdot)$ is continuous on $\mathcal{P}$. Thus $\mathcal{D}$ is an open set in $\mathcal{P}$. Since any potential φ in $\mathcal{D}$ has a unique Gibbs state it also has a unique equilibrium state by a theorem of Lanford and Ruelle (cf. Ref. 14, Theorem 4.2). It follows (Ref. 11, p. 96) that the pressure is continuously differentiable on $\mathcal{D}$. Our main result is the following.

Theorem 1. The pressure P is twice continuously differentiable on $\mathcal{D}$. Specifically, the second derivative

$$P''_\varphi(\psi_1, \psi_2) :\equiv \left. \frac{\partial^2 P(\varphi + u\psi_2 + v\psi_1)}{\partial u \, \partial v} \right|_{u=v=0}$$

exists for φ in $\mathcal{D}$ and ψ_1 and ψ_2 in $\mathcal{P}$, and, for fixed ψ_1 and ψ_2, is continuous in φ.

Moreover,

$$|P''_\varphi(\psi_1, \psi_2)| \leqslant 2[1 - \alpha(\varphi)]^{-2} \|\psi_1\| \, \|\psi_2\| \tag{2.7}$$

Remarks. 1. Dobrushin's uniqueness theorem is not only quite general but also quite strong. Simon[15] has shown that for any positive number ϵ there are multiple phase potentials within a $\mathcal{P}$ distance ϵ of $\mathcal{D}$. If there should actually be a critical point on the boundary of $\mathcal{D}$ then (2.7) could conceivably be used to get an estimate of a critical exponent by elaborating, in particular models, the dependence of $1 - \alpha(\varphi/kT)$ on T and on any external fields occurring in φ. But this is likely to be a crude estimate because of the generality of these techniques.

2. For an inequality going the other way from (2.7) see Ref. 2.

3. We have formulated the main result in a translation invariant context since this is the most interesting case and simplest to state. Moreover, the second derivative of the pressure (or free energy in some interpretations) has direct physical meaning (e.g., susceptibility). But the proof actually consists in showing that Gibbs states are continuously differentiable on $\mathcal{D}$. Thus translation invariance is not necessary for the key result of this paper.

3. SKETCH OF PROOF

Write $\Omega = X^{Z^m}$. Then Ω is a compact space and for any real-valued continuous function f on Ω put

$$(\tau_a f)(s) = \int f(x \vee t)\mu_a(dx \mid t) \tag{3.1}$$

for any point s in Ω, where t is the restriction of s to $Z^m - \{a\}$. $\tau_a f$ is again continuous. The operator τ_a depends of course on the potential φ. Enumerate the points of Z^m in any order and denote them $1, 2, 3, \ldots$. The operator

$$T_\varphi f = \lim_{n\to\infty} \tau_1\tau_2 \cdots \tau_n f \tag{3.2}$$

exists and the limit may be taken in the supremum norm in the space $C(\Omega)$ of all real-valued continuous functions on Ω. See, e.g., Ref. 6 for a discussion of convergence. If $\alpha(\varphi) < 1$ then φ has a unique Gibbs state, $\langle\ \rangle_\varphi$, and for any point $\bar{s}$ in Ω

$$\langle f\rangle_\varphi = \lim_{n\to\infty} (T_\varphi^n f)(\bar{s}) \tag{3.3}$$

for any function f in $C(\Omega)$. See Ref. 16, Section 6, or Ref. 6, Corollary 3.3, for a proof.

Informally, if we denote the derivative of T_φ in some direction ψ in $\mathcal{P}$ by T', i.e., $T' = dT_{\varphi+u\psi}/du$ at $u = 0$, and the derivative $d\langle f\rangle_{\varphi+u\psi}/du$ at $u = 0$ by $\langle f\rangle'$, then Eq. (3.3) gives

$$\langle f\rangle' = \lim_{n\to\infty} \sum_{k=0}^{n-1} (T^{n-k-1}T'T^k f)(\bar{s}) \tag{3.4}$$

where we have written simply T for T_φ. But using Eq. (3.3) again for each term in the sum yields

$$\langle f\rangle' = \sum_{k=0}^{\infty} \langle T'T^k f\rangle_\varphi$$

In the next section we shall justify these steps and show that the series converges. Of course $\langle f\rangle'$ represents a second derivative of the pressure [see Eq. (4.22)]. Finally, performing the sum on k yields

$$\langle f\rangle' = \langle T'(1-T)^{-1} f\rangle_\varphi \tag{3.5}$$

We shall see that this is meaningful and correct if $1 - T$ is interpreted as an operator in a suitable space of functions modulo constant functions.

Since the equilibrium state $\langle\ \rangle_\varphi$ determines the first derivative of the pressure (see, e.g., Ref. 10, Section II.1) we get, finally, for the second derivative of the pressure in the directions ψ_1, ψ

$$P''_\varphi(\psi_1, \psi) = \langle T'(1-T)^{-1} f_{\psi_1}\rangle_\varphi \tag{3.6}$$

where f_{ψ_1} is a function on Ω linearly associated to ψ_1 [see Eq. (4.23)].

4. PROOFS

The space $\Omega \equiv X^{Z^m}$ is a compact space on which we shall consider various classes of functions. For any function f in the space $C(\Omega)$ of continuous real-valued functions on Ω put, for j in Z^m,

$$\delta_j(f) = \sup\{|f(s) - f(t)| : s, t \in \Omega, \quad s = t \text{ except at } j\}$$

and define

$$|f|_1 = \sum_{j \in Z^m} \delta_j(f)$$

C^1 will denote the space of functions f in $C(\Omega)$ for which (a) $|f|_1 < \infty$ and (b) there exists a sequence f_n of continuous cylinder functions on Ω such that $|f - f_n|_1 \to 0$ as $n \to \infty$.

The kernel of the norm $|\ |_1$ consists exactly of the constant functions on Ω. The quotient space is a Banach space.

For any potential φ in $\mathcal{P}$ we also put

$$\|\varphi\|_1 = \sum_{A \ni 0} \sup\{|\varphi(s)| : s \in X^A\}$$

and, for $a \neq 0$ in Z^m,

$$\zeta_a(\varphi) = \sup\left\{\Big|\sum_{A \supset \{0,a\}} \varphi(s|A)\Big| : s \in \Omega\right\}$$

$\zeta_a(\varphi)$ measures the "decay rate" of the many-body potential φ. Note that $\|\varphi\|_1 \leqslant \|\varphi\|$.

When the dependence of the conditional probabilities μ_a [cf. (2.4)] on φ must be exhibited we write μ_a^φ.

The two inequalities of the following lemma are the key ingredients of the proof.

Lemma 1. Let φ and ψ be in $\mathcal{P}$ and let h be in $C(X)$. Let s and s' be two points of Ω which differ only at the point $a \neq 0$ in Z^m. Then

$$|\partial(\mu_0^{\varphi+u\psi}h)(s)/\partial u|_{u=0}| \leqslant \|\psi\|_1 \delta(h) \tag{4.1}$$

where $\delta(h) = \sup h - \inf h$ and

$$\begin{aligned} &|\partial(\mu_0^{\varphi+u\psi}h)(s)/\partial u|_{u=0} - \partial(\mu_0^{\varphi+u\psi}h)(s')/\partial u|_{u=0}| \\ &\quad \leqslant 2\zeta_a(\psi)\min(|h|_\infty, \delta(h)) + 2\|\psi\|_1\zeta_a(\varphi)\delta(h) \end{aligned} \tag{4.2}$$

Proof. Let t and t' be the restrictions of s and s', respectively, to $Z^m - \{0\}$. Put

$$g(x) = -\sum_{A \ni 0} \varphi(x \vee t \,|\, A), \qquad g'(x) = -\sum_{A \ni 0} \varphi(x \vee t' \,|\, A)$$

and

$$k(x) = -\sum_{A \ni 0} \psi(x \vee t \,|\, A), \qquad k'(x) = -\sum_{A \ni 0} \psi(x \vee t' \,|\, A)$$

Then by the definition (2.4) we have

$$(\mu_0^{\varphi+u\psi}h)(s) = \left[\int_X e^{g(x)+uk(x)} d\nu(x)\right]^{-1} \int_X e^{g(x)+uk(x)} h(x)\, d\nu(x) \tag{4.3}$$

Since g and k are continuous on X, and *a fortiori* bounded, one may conclude easily that

$$\partial(\mu_0^{\varphi+u\psi}h)(s)/\partial u|_{u=0} = \langle (k - \langle k\rangle_g)h\rangle_g \tag{4.4}$$

where $\langle\ \rangle_g$ denotes expectation with respect to the normalized measure $e^g\, d\nu / \int e^g\, d\nu$.

The right side is clearly zero if h is a constant function on X. Thus the right side is unaffected by subtracting $\langle h\rangle_g$ from h. Hence

$$\begin{aligned} |\partial(\mu_0^{\varphi+u\psi}h)(s)/\partial u|_{u=0}| &\leqslant |\langle (k - \langle k\rangle_g)(h - \langle h\rangle_g)\rangle_g| \\ &\leqslant \|k - \langle k\rangle_g\|_{L^2} \|h - \langle h\rangle_g\|_{L^2} \\ &\leqslant \|k\|_{L^2} \|h - \langle h\rangle_g\|_{L^2} \\ &\leqslant |k|_\infty (\sup h - \inf h) \end{aligned}$$

In these inequalities the L^2 norms are computed with respect to $\langle\ \rangle_g$. Since $|k|_\infty \leqslant \|\psi\|_1$ inequality (4.1) is proved.

Now put $g_v = vg + (1-v)g'$ and $k_v = vk + (1-v)k'$ for $0 \leqslant v \leqslant 1$. Let $F(v) = \langle (k_v - \langle k_v \rangle_{g_v}) h \rangle_{g_v}$. Then by (4.4) the left side of (4.2) is exactly $|F(1) - F(0)|$, which we estimate as in Ref. 15 by writing

$$F(1) - F(0) = \int_0^1 (dF/dv)\, dv$$

Fix a number v_0 in $[0,1]$. $F(v)$ does not change if we subtract a constant from h. So we subtract $\langle h \rangle_{g_{v_0}}$. In view of (4.3) we have, writing $\langle\ \rangle_v$ in place of $\langle\ \rangle_{g_v}$,

$$\begin{aligned}
\left.\frac{dF}{dv}\right|_{v=v_0} &= \left.\frac{d}{dv}\langle (k_v - \langle k_v \rangle_v)(h - \langle h \rangle_{v_0}) \rangle_v \right|_{v=v_0} \\
&= \langle (k - k' - \langle k - k' \rangle_{v_0})(h - \langle h \rangle_{v_0}) \rangle_{v_0} - \frac{d}{dv}\langle k_{v_0} \rangle_v \langle h - \langle h \rangle_{v_0} \rangle_{v_0} \\
&\quad + \langle [g - g' - \langle g - g' \rangle_{v_0}][k_{v_0} - \langle k_{v_0} \rangle_{v_0}][h - \langle h \rangle_{v_0}] \rangle_{v_0}
\end{aligned}$$

The second factor of the second term is zero. The first and third terms may be estimated using the Schwartz inequality for the expectation $\langle\ \rangle_{v_0}$, giving

$$\begin{aligned}
\left|\frac{dF}{dv}\right| &\leqslant \|k - k' - \langle k - k' \rangle_{v_0}\|_{L^2} \|h - \langle h \rangle_{v_0}\|_{L^2} \\
&\quad + \|g - g' - \langle g - g' \rangle_{v_0}\|_{L^2} \|k_{v_0} - \langle k_{v_0} \rangle_{v_0}\|_{L^2} \sup |h - \langle h \rangle_{v_0}|
\end{aligned}$$

Since $\|h - \langle h \rangle_{v_0}\|_{L^2}$ is dominated by both $|h|_\infty$ and $\delta(h)$ we have

$$\left|\frac{dF}{dv}\right| \leqslant |k - k'|_\infty \min(|h|_\infty, \delta(h)) + |g - g'|_\infty |k_{v_0}|_\infty \delta(h)$$

But $|k_v|_\infty \leqslant \|\psi\|_1$ for all v in $[0,1]$, while

$$\begin{aligned}
|g(x) - g'(x)| &= \Big| \sum_{A \ni 0} [\varphi(x \vee t | A) - \varphi(x \vee t' | A)] \Big| \\
&= \Big| \sum_{A \supset \{0,a\}} [\phi(x \vee t | A) - \varphi(x \vee t' | A)] \Big| \\
&\leqslant 2\zeta_a(\varphi)
\end{aligned}$$

Thus

$$\left|\frac{dF}{dv}\right| \leqslant 2\zeta_a(\psi) \min(|h|_\infty, \delta(h)) + 2\zeta_a(\varphi)\|\psi\|_1 \delta(h)$$

Integrating this inequality from 0 to 1 yields (4.2) and concludes the lemma.

Proposition 2. $\alpha(\cdot)$ is continuous on $\mathcal{P}$. In fact,

$$|\alpha(\varphi') - \alpha(\varphi)| \leqslant (1 + \|\varphi\| + \|\varphi'\|)\|\varphi' - \varphi\| \tag{4.5}$$

Proof. Write $\rho_b(\varphi)$ for the expression in (2.5). Then

$$|\alpha(\varphi') - \alpha(\varphi)| \leqslant \sum_{a\neq 0} |\rho_a(\varphi') - \rho_a(\varphi)|$$
$$\leqslant (1/2) \sum_{a\neq 0} \sup\{\| \mu_0^{\varphi'}(\ |t) - \mu_0^{\varphi'}(\ |t')\|_{\mathrm{Var}}$$
$$- \| \mu_0^{\varphi}(\ |t) - \mu_0^{\varphi}(\ |t')\|_{\mathrm{Var}} : t = t' \quad \text{off } \{a\}\}$$
$$\leqslant (1/2) \sum_{a\neq 0} \sup\{\| \mu_0^{\varphi'}(\ |t) - \mu_0^{\varphi'}(\ |t')$$
$$-[\mu_0^{\varphi}(\ |t) - \mu_0^{\varphi}(\ |t')]\|_{\mathrm{Var}} : t = t' \quad \text{off } \{a\}\}$$

But, putting $\psi = \varphi' - \varphi$ and $\varphi_u = \varphi + u\psi$, we have

$$\| \mu_0^{\varphi'}(\ |t) - \mu_0^{\varphi'}(\ |t') - (\mu_0^{\varphi}(\ |t) - \mu_0^{\varphi}(\ |t'))\|_{\mathrm{Var}}$$
$$= \sup_{h\in C(X), |h|_\infty \leqslant 1} |\mu_0^{\varphi'}(h)(t) - \mu_0^{\varphi}(h)(t) - [\mu_0^{\varphi'}(h)(t') - \mu_0^{\varphi}(h)(t')]|$$
$$= \sup_{|h|_\infty \leqslant 1} \left| \int_0^1 \left[\frac{\partial \mu_0^{\varphi_u}(h)(t)}{\partial u} - \frac{\partial \mu_0^{\varphi_u}(h)(t')}{\partial u} \right] du \right|$$
$$\leqslant \sup_{|h|_\infty = 1} \int_0^1 [2\zeta_a(\psi)|h|_\infty + 2\|\psi\|_1 \zeta_a(\varphi_u)\delta(h)]\, du$$

by Lemma 1. But $\delta(h) \leqslant 2|h|_\infty$ and $\zeta_a(\varphi_u) \leqslant u\zeta_a(\varphi') + (1-u)\zeta_a(\varphi)$ since ζ_a is a seminorm. Thus

$$|\alpha(\varphi') - \alpha(\varphi)| \leqslant \sum_{a\neq 0} \int_0^1 [\zeta_a(\psi) + 2\|\psi\|_1 (u\zeta_a(\varphi') + (1-u)\zeta_a(\varphi))]\, du$$
$$\leqslant \sum_{a\neq 0} [\zeta_a(\psi) + \|\psi\|_1 (\zeta_a(\varphi') + \zeta_a(\varphi))]$$

But

$$\sum_{a\neq 0} \zeta_a(\varphi) \leqslant \sum_{a\neq 0} \sum_{A\supset\{0,a\}} \sup\{|\varphi(s)| : s \in X^A\}$$
$$\leqslant \sum_{A\ni 0} (|A| - 1) \sup\{|\varphi(s)| : s\in X^A\}$$
$$\leqslant \|\varphi\| \tag{4.6}$$

and of course $\|\psi\|_1 \leqslant \|\psi\|$. The proposition now follows from this and the last estimate of $|\alpha(\varphi') - \alpha(\varphi)|$. Note incidentally that since $\alpha(0) = 0$, $\alpha(\cdot)$ is actually finite on all of $\mathcal{P}$, with $\alpha(\varphi) \leqslant (1 + \|\varphi\|)\|\varphi\|$, as follows by putting $\varphi' = 0$ in all the previous inequalities. ■

Lemma 3. If f is in C^1 and φ is in $\mathcal{P}$ then $\tau_j^\varphi f$ is in C^1 and the map $\varphi \to \tau_j^\varphi f$ is continuous from $\mathcal{P}$ into C^1 for any point j in Z^m.

Proof. We may take $j = 0$ without loss of generality. Suppose that $a \neq 0$ is in Z^m and that s and s' are two configurations in Ω which differ only at a. If f is any function in $C(\Omega)$ then, writing t and t' for the restrictions of s and s', respectively, to $Z^m - \{0\}$, we have

$$\begin{aligned}(\tau_0^\varphi f)(s) - (\tau_0^\varphi f)(s') &= \int f(x,t)\mu_0^\varphi(dx\,|\,s) - \int f(x,t')\mu_0^\varphi(dx\,|\,s') \\ &= \int \left[f(x,t) - f(x,t') \right] \mu_0^\varphi(dx\,|\,s) \\ &\quad + \int f(x,t')\left[\mu_0^\varphi(dx\,|\,s) - \mu_0^\varphi(dx\,|\,s') \right] \\ &= A(\varphi) + B(\varphi) \end{aligned} \tag{4.7}$$

where $A(\varphi)$ and $B(\varphi)$ are the values of the two integrals. Note first that $|A(\varphi)| \leqslant \sup_x |f(x,t) - f(x,t')| \leqslant \delta_a(f)$. Moreover, since the signed measure in brackets in $B(\varphi)$ has total mass zero we may subtract $[\sup_x f(x, t') + \inf_x f(x,t')]/2$ from the integrand in $B(\varphi)$ yielding

$$|B(\varphi)| \leqslant (1/2)\delta_0(f)\|\mu_0^\varphi(\cdot\,|\,s) - \mu_0^\varphi(\cdot\,|\,s')\|_{\mathrm{Var}} \leqslant \delta_0(f)\rho_a(\varphi)$$

Thus $\delta_a(\tau_0^\varphi f) \leqslant \delta_a(f) + \delta_0(f)\rho_a(\varphi)$. Summing on a yields [note that $\delta_0(\tau_0^\varphi f) = 0$]

$$\|\tau_0^\varphi f\|_1 \leqslant |f|_1 + \delta_0(f)\alpha(\varphi) \tag{4.8}$$

Now from (4.1) we have

$$\begin{aligned}\left| \frac{\partial A(\varphi + u\psi)}{\partial u} \right| &\leqslant \|\psi\|_1 \delta(f(\cdot, t) - f(\cdot, t')) \\ &\leqslant 2\|\psi\|_1 |f(\cdot,t) - f(\cdot,t')|_\infty \\ &\leqslant 2\|\psi\|_1 \delta_a(f)\end{aligned}$$

while from (4.2) we have

$$\left| \frac{\partial B(\varphi + u\psi)}{\partial u} \right| \leqslant 2\zeta_a(\psi)\delta_0(f) + 2\|\psi\|_1 \zeta_a(\varphi + u\psi)\delta_0(f)$$

Hence

$$\begin{aligned}&\left|(\tau_0^{\varphi+\psi} f)(s) - (\tau_0^\varphi f)(s) - \left[(\tau_0^{\varphi+\psi} f)(s') - (\tau_0^\varphi f)(s')\right]\right| \\ &\quad \leqslant \int_0^1 |\partial(\tau_0^{\varphi+u\psi} f)(s)/\partial u - \partial(\tau_0^{\varphi+u\psi} f)(s')/\partial u|\, du \\ &\quad \leqslant 2\int_0^1 \left[\|\psi\|_1 \delta_a(f) + (\zeta_a(\psi) + \|\psi\|_1 \zeta_a(\varphi + u\psi))\delta_0(f) \right] du\end{aligned}$$

Since $\zeta_a(\varphi + u\varphi) \leqslant \zeta_a(\varphi) + u\zeta_a(\psi)$ we may estimate the integral and, using (4.6), sum on a to get

$$\|\tau_0^{\varphi+\psi}f - \tau_0^{\varphi}f\|_1 \leqslant 2\|\psi\|_1|f|_1 + 2\big[\|\psi\| + \|\psi\|_1(\|\varphi\| + (1/2)\|\psi\|)\big]\delta_0(f) \tag{4.9}$$

Suppose now that φ is a finite-range interaction and that f is a continuous function on Ω depending on only finitely many coordinates. Then we see from Eqs. (2.2)–(2.4) that the measure $\mu_0(\cdot\,|\,s)$ depends on only finitely many coordinates of s and, by (3.1), so does $\tau_0^{\varphi}f$. If f is now any function in C^1 and f_n are continuous cylinder functions converging to f in C^1 sense then (4.8) shows that $\|\tau_0^{\varphi}f - \tau_0^{\varphi}f_n\|_1 \to 0$ so that $\tau_0^{\varphi}f$ is in C^1 if φ is of finite range. Finally if φ is an arbitrary interaction in $\mathcal{P}$ and φ_n is a sequence of finite-range interactions converging to φ in $\mathcal{P}$ norm, then for any f in C^1, (4.9) shows that $|\tau_0^{\varphi_n}f - \tau_0^{\varphi}f|_1$ goes to zero as $n \to \infty$. Hence $\tau_0^{\varphi}f$ is in C^1 for any φ in $\mathcal{P}$ and f in C^1. Moreover (4.9) also shows that $\varphi \to \tau_0^{\varphi}f$ is continuous from $\mathcal{P}$ into C^1.

Lemma 4. For any potentials φ and ψ in $\mathcal{P}$ and any function f in $C(\Omega)$ the function $u \to \tau_j^{\varphi+u\psi}f$ from the real line into the Banach space $C(\Omega)$ is differentiable, and its derivative

$$D_\psi \tau_j^{\varphi} f \equiv d\tau_j^{\varphi+u\psi}f/du|_{u=0}$$

is continuous in φ as a function from $\mathcal{P}$ into $C(\Omega)$. Moreover

$$|D_\psi \tau_j^{\varphi} f|_\infty \leqslant \delta_j(f)\|\psi\|_1 \tag{4.10}$$

Proof. From the definitions of (2.2)–(2.4) and (3.1) we have

$$\begin{aligned}(\tau_j^{\varphi+u\psi}f)(s) = \int_X &\exp\big[-W_j^{\varphi}(x,t) - uW_j^{\psi}(x,t)\big] \\ &\times f(x,t)\nu(dx)\Big/\int_X \exp\big[-W_j^{\varphi+u\psi}(x,t)\big]\nu(dx)\end{aligned}$$

where $t = s$ restricted to $Z^m - \{j\}$. Since all of the functions W_j and f are continuous functions of all arguments and *a fortiori* bounded, it is clear that the right side is continuous in s, is continuous in u as a function into $C(\Omega)$, and, moreover, since the difference quotient for the exponential function converges uniformly to its derivative on bounded sets, the difference quotient for $u \to \tau_j^{\varphi+u\psi}f$ converges uniformly in s to its derivative, which at $u = 0$ is given by

$$\begin{aligned}(D_\psi \tau_j^{\varphi} f)(s) = &-\int_X \exp\big[-W_j^{\varphi}(x,t)\big]W_j^{\psi}(x,t)f(x,t)\nu(dx)/Z_j^{\varphi}(t) \\ &+\int_X \exp\big[-W_j^{\varphi}(x,t)\big]W_j^{\psi}(x,t)\nu(dx) \\ &\times\int \exp\big[-W_j^{\varphi}(x,t)\big]f(x,t)\nu(dx)/\big[Z_j^{\varphi}(t)\big]^2\end{aligned}$$

Since the map $\varphi \to W_j^\varphi(s)$ is continuous from $\mathcal{P}$ into $C(\Omega)$ it is clear that so is the map $\varphi \to D_\psi \tau_j^\varphi f$. Finally, taking into account the translation invariance of the $\|\psi\|_1$ norm the inequality (4.10) is just the inequality (4.1) with $h(x) = f(x, t)$. ■

Recall the definition (3.2), of the operator $T_\varphi : C(\Omega) \to C(\Omega)$ corresponding to a potential φ in $\mathcal{P}$. A discussion of this operator, first used by L. N. Vasershtein,[16] appears in Ref. 6 along with some related inequalities which we shall need.

Lemma 5. Let f be a function in C^1, let φ be in $\mathcal{D}$, and let ψ be in $\mathcal{P}$. Then the function $u \to T_{\varphi + u\psi} f$ from the real line to $C(\Omega)$ is differentiable at $u = 0$. Its derivative,

$$D_\psi T_\varphi f \equiv dT_{\varphi + u\psi} f / du|_{u=0} \tag{4.11}$$

is a continuous function of φ from $\mathcal{D}$ to $C(\Omega)$ and

$$|D_\psi T_\varphi f|_\infty \leqslant [1 - \alpha(\varphi)]^{-1} \|\psi\|_1 |f|_1 \tag{4.12}$$

Proof. Informally, the chain rule shows

$$D_\psi T_\varphi f = \sum_{j=1}^{\infty} \tau_1^\varphi \cdots \tau_{j-1}^\varphi (D_\psi \tau_j^\varphi) \tau_{j+1}^\varphi \cdots f \tag{4.13}$$

We shall show first that the series actually converges uniformly on Ω. For in fact since each τ_k^φ is a contraction in sup norm

$$\begin{aligned}
\sum_{j=1}^{\infty} |\tau_1^\varphi \cdots \tau_{j-1}^\varphi (D_\psi \tau_j^\varphi) \tau_{j+1}^\varphi \cdots f|_\infty &\leqslant \sum_{j=1}^{\infty} |(D_\psi \tau_j^\varphi) \tau_{j+1}^\varphi \cdots f|_\infty \\
&\leqslant \sum_{j=1}^{\infty} \|\psi\|_1 \delta_j (\tau_{j+1}^\varphi \cdots f) \\
&\leqslant \|\psi\|_1 |f|_1 [1 - \alpha(\varphi)]^{-1}
\end{aligned}$$

wherein we have used (4.10) in the second line and inequality (3.26) of Ref. 6 (with $d = 0$) in the third line.

Write $A_\varphi f$ for the right side of (4.13). Since the series converges uniformly on Ω, as these inequalities show, and since each term is a continuous function on Ω, $A_\varphi f$ is in $C(\Omega)$. Moreover,

$$|A_\varphi f|_\infty \leqslant \|\psi\|_1 |f|_1 [1 - \alpha(\varphi)]^{-1} \tag{4.14}$$

Now if f depends on only finitely many coordinates then $\tau_k^\varphi f = f$ for all sufficiently large k, and of course $D_\psi \tau_k^\varphi f = 0$ for such k. Since $T_\varphi f$ involves, in this case, only a finite product of operators, the chain rule is indeed applicable, Eq. (4.13) is correct in view of Lemma 4, and the right side is a finite sum. Thus

$$\left.\frac{dT_{\varphi+u\psi}f}{du}\right|_{u=0} = A_\varphi f \tag{4.15}$$

if f is a continuous cylinder function. Moreover, the map $\varphi \to A_\varphi f$ is continuous on $\mathfrak{D}$ to $C(\Omega)$, in this case by Lemma 4, since $A_\varphi f$ is a polynomial in continuously φ-dependent operators acting on f. But if f is an arbitrary function in C^1 there is a sequence, f_n, of continuous cylinder functions such that $|f_n - f|_1 \to 0$. Thus by (4.14) $|A_\varphi f_n - A_\varphi f|_\infty$ converges to zero uniformly on each set $\{\varphi \in \mathfrak{P} : \alpha(\varphi) \leqslant \kappa < 1\}$. It follows first that the map $\varphi \to A_\varphi f$ is continuous from $\mathfrak{D}$ into $C(\Omega)$. Secondly it follows that for each φ in the open set $\mathfrak{D}$ that $dT_{\varphi+u\psi}f_n/du$ converges in $C(\Omega)$ norm, uniformly for u in a neighborhood of zero, to $A_{\varphi+u\psi}f$. Since the latter is a continuous function of u into $C(\Omega)$ it follows from an elementary theorem on uniform convergence of derivatives that $T_{\varphi+u\psi}f$ is actually differentiable as a function of u into $C(\Omega)$ for small u, and that its derivative is $A_{\varphi+u\psi}f$. Thus (4.15) holds for all f in C^1 and (4.12) follows from (4.14). ■

Lemma 6. Let $\langle\ \rangle_\varphi$ denote expectation with respect to the unique Gibbs state corresponding to the potential φ in $\mathfrak{D}$. If f is in C^1 then $\langle f\rangle_\varphi$ is a continuously differentiable function of φ in $\mathfrak{D}$, in the sense that

$$D_\psi\langle f\rangle_\varphi \equiv d\langle f\rangle_{\varphi+u\psi}/du|_{u=0}$$

exists for each φ in $\mathfrak{D}$ and ψ in $\mathfrak{P}$, and, for fixed ψ, is a continuous function of φ. Moreover,

$$|D_\psi\langle f\rangle_\varphi| \leqslant [1-\alpha(\varphi)]^{-2}\|\psi\|_1|f|_1 \tag{4.16}$$

The derivative is given by the convergent series

$$D_\psi\langle f\rangle_\varphi = \sum_{k=1}^{\infty}\langle(D_\psi T_\varphi)T_\varphi^{k-1}f\rangle_\varphi \tag{4.17}$$

Proof. Choose a point $\bar{s}$ in Ω. For φ in $\mathfrak{D}$ and any function g in $C(\Omega)$ we have

$$\langle g\rangle_\varphi = \lim_{n\to\infty}(T_\varphi^n g)(\bar{s}) \tag{4.18}$$

This is essentially Dobrushin's basic result. (See, e.g., Corollary 3.3 of Ref. 6 for a proof.)

Let us remark that as a function of φ, $T_\varphi g$ is continuous from $\mathfrak{D}$ into $C(\Omega)$ if g is in $C(\Omega)$, and is continuous from $\mathfrak{D}$ into C^1 if g is in C^1. Both of these assertions can be proved by observing first that they are true if g is a continuous cylinder function, since in this case $T_\varphi g = \tau_1^\varphi\tau_2^\varphi \cdots \tau_p^\varphi g$ for some p, so that Lemma 3 may be applied p times. Second, if g is an arbitrary function in C^1 and g_n are continuous cylinder functions with $|g_n - g|_1 \to 0$, then $|T_\varphi g_n - T_\varphi g|_1 \leqslant \alpha(\varphi)|g_n - g|_1 \leqslant |g_n - g|_1$ for all φ in $\mathcal{D}$. Hence $T_\varphi g_n$ converges uniformly on $\mathcal{D}$ to $T_\varphi g$, showing that $\varphi \to T_\varphi g$ is continuous from $\mathcal{D}$ to C^1. A similar argument shows $\varphi \to T_\varphi g$ is continuous from $\mathcal{P}$ to $C(\Omega)$ when g is in $C(\Omega)$ since T_φ is a contraction in $C(\Omega)$ norm for all φ.

Now let f be in C^1. Informally, we have by the chain rule

$$d\left[(T_{\varphi+u\psi})^n f\right](\bar{s})/du|_{u=0} = \sum_{k=1}^{n} \left(T_\varphi^{n-k} T_\varphi' T_\varphi^{k-1} f\right)(\bar{s}) \tag{4.19}$$

where $T_\varphi' g = dT_{\varphi+u\psi} g/du$ at $u = 0$. This step is easily justified in view of Lemma 5 and the preceding remark since

$$(T_{\varphi+u\psi})^n f - (T_\varphi)^n f = \sum_{k=1}^{n} (T_{\varphi+u\psi})^{n-k} (T_{\varphi+u\psi} - T_\varphi)(T_\varphi)^{k-1} f$$

so that the difference quotient converges uniformly on Ω as $u \to 0$. One must use here the contractivity of $(T_{\varphi+u\psi})^{n-k}$ in $C(\Omega)$ and its strong continuity in u as well as a standard double approximation argument.

Using (4.18) we have

$$\lim_{n\to\infty} \left(T_\varphi^{n-k} T_\varphi' T_\varphi^{k-1} f\right)(\bar{s}) = \langle T_\varphi' T_\varphi^{k-1} f\rangle_\varphi .$$

Moreover,

$$\begin{aligned} |\left(T_\varphi^{n-k} T_\varphi' T_\varphi^{k-1} f\right)(\bar{s})| &\leqslant |T_\varphi' T_\varphi^{k-1} f|_\infty \\ &\leqslant \left[1 - \alpha(\varphi)\right]^{-1} \|\psi\|_1 |T_\varphi^{k-1} f|_1 \\ &\leqslant \left[1 - \alpha(\varphi)\right]^{-1} \alpha(\varphi)^{k-1} \|\psi\|_1 |f|_1 \end{aligned} \tag{4.20}$$

Therefore, since $\alpha(\varphi) < 1$, the limit of the right side of (4.19) exists and converges for each φ in $\mathcal{D}$ to

$$\sum_{k=1}^{\infty} \langle T_\varphi' T_\varphi^{k-1} f\rangle_\varphi \tag{4.21}$$

Since the kth term is majorized by $[1 - \alpha(\varphi)]^{-1} \alpha(\varphi)^{k-1} \|\psi\|_1 |f|_1$ the sum of the series is no more than $[1 - \alpha(\varphi)]^{-2} \|\psi\|_1 |f|_1$.

This would establish (4.16) and (4.17) if we knew that the series actually represents the derivative on $\mathcal{D}$. To this end it suffices to show that the right side of (4.19) converges to (4.21) pointwise and boundedly on each open set $A_\kappa = \{\varphi : \alpha(\varphi) < \kappa < 1\}$, and that the sum (4.21) is continuous on $\mathcal{D}$. For then the dominated convergence theorem yields for φ in $\mathcal{D}$ and small v, $\langle f\rangle_{\varphi+v\psi} - \langle f\rangle_\varphi = \lim(T_{\varphi+v\psi}^n f)(\bar{s}) - \lim(T_\varphi^n f)(\bar{s}) = \lim\int_0^v [d(T_{\varphi+u\psi}^n f)(\bar{s})/du]\,du = \int_0^v F(\varphi + u\psi)\,du$, where $F(\varphi)$ denotes the sum (4.21). The fundamental theorem of calculus can then be applied. But the boundedness of the convergence on A_κ follows from the inequalities (4.20), and these same estimates show that (4.21) will be continuous on each set A_κ, and hence on $\mathfrak{D}$, if each term $\langle T'_\varphi T_\varphi^{k-1} f\rangle_\varphi$ is a continuous function on A_κ. But the map $\varphi \to T_\varphi^{k-1} f$ is continuous on $\mathfrak{D}$ to C^1 as remarked early in the proof of this lemma. Moreover the map $\varphi \to T'_\varphi g$ is continuous from A_κ to $C(\Omega)$ for g in C^1 by Lemma 5 and $|T'_\varphi g|_\infty \leqslant (1-\kappa)^{-1}\|\psi\|_1 |g|_1$ for all φ in A_κ. It follows that the map $\varphi \to T'_\varphi T_\varphi^{k-1} f$ is continuous on A_κ to $C(\Omega)$. Thus the lemma will be concluded once it is shown that for fixed g in $C(\Omega)$ the map $\varphi \to \langle g\rangle_\varphi$ is continuous on each A_κ. But the proof of Corollary 3.3 of Ref. 6 shows that for fixed g in C^1 $(T_\varphi^n g)(\bar{s})$ converges uniformly to $\langle g\rangle_\varphi$ on A_κ. So $\langle g\rangle_\varphi$ is continuous in φ for g in C^1. But since C^1 is dense in $C(\Omega)$ and the linear functionals $\langle\cdot\rangle_\varphi$ are uniformly bounded on $C(\Omega)$ the continuity $\varphi \to \langle g\rangle_\varphi$ for g in $C(\Omega)$ follows. This concludes the proof of the lemma. ■

Proof of Theorem. A potential φ in the Dobrushin uniqueness region $\mathfrak{D}$ has a unique Gibbs state by the Dobrushin uniqueness theorem. Since every equilibrium state is also a Gibbs state (cf. Ref. 14, Theorem 4.2, or Ref. 10, Theorem III, 2.1), φ has a unique equilibrium state. That is to say, the pressure, as a convex function on $\mathfrak{P}$, has a unique tangent functional. This implies that the pressure P has a derivative, $\partial P(\varphi + u\psi)/\partial u|_{u=0}$, as noted in Ref. 11, p. 96 (or see Ref. 12, Theorem 44A, p. 113). Moreover the derivative is given by (cf. Ref. 10, Section II.1)

$$D_\psi P(\varphi) \equiv \partial P(\varphi + u\psi)/\partial u|_{u=0} = \langle f_\psi\rangle_\varphi \tag{4.22}$$

where

$$f_\psi(s) = -\sum_{A \ni 0} |A|^{-1}\psi(s\,|\,A) \tag{4.23}$$

Now f_ψ is in C^1 when ψ is in $\mathfrak{P}$, and in fact even when $\|\psi\|_1 < \infty$. For if s and t are configurations which differ only at a point j in Z^m, then

$$\begin{aligned}
|f_\psi(s) - f_\psi(t)| &= \Big|\sum_{A \ni 0} |A|^{-1}[\psi(s\,|\,A) - \psi(t\,|\,A)]\Big| \\
&= \Big|\sum_{A \supset \{0,j\}} |A|^{-1}[\psi(s\,|\,A) - \psi(t\,|\,A)]\Big| \\
&\leqslant 2 \sum_{A \supset \{0,j\}} |A|^{-1} \sup_{u \in \Omega} |\psi(u\,|\,A)|
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{j \in Z^m} \delta_j(f_\psi) &\leqslant 2 \sum_j \sum_{A \supset \{0,j\}} |A|^{-1} \sup_{u \in \Omega} |\psi(u\,|\,A)| \\
&= 2 \sum_{A \ni 0} \sup_{u \in \Omega} |\psi(u\,|\,A)| \\
&= 2\|\psi\|_1
\end{aligned}$$

Thus

$$|f_\psi|_1 \leqslant 2\|\psi\|_1 \tag{4.24}$$

But by Lemma 6 $\langle f_\psi \rangle_\varphi$ is a differentiable function of φ. Thus P is twice differentiable and

$$P_\varphi''(\psi_1, \psi_2) \equiv \left. \frac{\partial^2 P(\varphi + u\psi_2 + v\psi_1)}{\partial u\, \partial v} \right|_{u=v=0}$$
$$= D_{\psi_2} \langle f_{\psi_1} \rangle_\varphi$$

By Lemma 6 the right side is continuous in φ for fixed ψ_1 and ψ_2, and moreover

$$|P_\varphi''(\psi_1, \psi_2)| \leqslant [1 - \alpha(\varphi)]^{-2} \|\psi_2\|_1 |f_{\psi_1}|_1$$
$$\leqslant 2[1 - \alpha(\varphi)]^{-2} \|\psi_2\|_1 \|\psi_1\|_1$$

which is a slightly stronger inequality than (2.7) because $\|\psi\|_l \leqslant \|\psi\|$.

Remark. The equation (3.6) can now be easily interpreted. By Eqs. (4.17) and (4.22) we have

$$P_\varphi''(\psi_1, \psi) = \sum_{k=1}^{\infty} \langle T_\varphi' T_\varphi^{k-1} f_{\psi_1} \rangle_\varphi \tag{4.25}$$

where f_{ψ_1} is given by Eq. (4.23) and T_φ' is $D_\psi T_\varphi$, as defined in (4.11) and discussed in Lemma 5. But since T_φ takes constant functions to constant functions we may regard T_φ as an operator on the Banach space of C^1 functions modulo constant functions. On this space we have $\|T_\varphi\| \leqslant \alpha(\varphi) < 1$ for φ in $\mathfrak{D}$. Hence $1 - T_\varphi$ has an inverse on this quotient space and $(1 - T_\varphi)^{-1} = \sum_{k=1}^{\infty} T_\varphi^{k-1}$. Now T_φ' annihilates constants because $T_\varphi 1$ is independent of φ. Thus T_φ' may be regarded as a map from the quotient space to functions. Lemma 5 shows that this map is continuous from C^1 norm to $C(\Omega)$. Thus $T_\varphi'(1 - T_\varphi)^{-1}$ is a continuous and well-defined map from C^1/constants into $C(\Omega)$. It may be applied to a function f in C^1 by first projecting f into the quotient space. With this interpretation of $(1 - T_\varphi)^{-1}$ the right side of (4.25) may be written $\langle T_\varphi'(1 - T_\varphi)^{-1} f_{\psi_1} \rangle_\varphi$. This makes clear the meaning of Eq. (3.6).

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